3. Birikh, R. V., On small perturbations of a plane parallel flow with cubic velocity profile. PMM Vol. 30, №2, 1966.
4. Rudakov, R. N., Spectrum of perturbations and stability of convective motion between vertical planes. PMM Vol. 31, №2, 1967.
5. Birikh, R. V., Gershuni, G. Z., Zhukhovitskii, E. M. and Rudakov, R. N., Hydrodynamic and thermal instability of a steady convective flow. PMM Vol. 32, №2, 1968.
6. Ostroumov, G. A., Free Convection Under the Conditions of the Interior Problem. Moscow-Leningrad, Gostekhizdat, 1952.
7. Ostrach, S., On the flow, heat transfer, and stability of viscous fluids subject to body forces and heated from below in vertical channels. 50 Jahre Grenzschicht forsch., Berlin, Acad. Verl., 1956.
8. Zaitsev, V. M. and Sorokin, M. P., On the stability of thermal convective motion of a fluid in a vertical slot. Uch. Zap. Permsk. Univ. Vol. 19, №3, 1961.
9. Vest, C. M. and Arpaci, V. S., Stability of natural convection in a vertical slot. J. Fluid Mech., Vol. 36, p. 1, 1969.
10. Gershuni, G. Z., Zhukhovitskii, E. M. and Rudakov, R. N., On the theory of Rayleigh instability. PMM Vol. 31, №5, 1967.
11. Gershuni, G. Z., Zhukhovitskii, E. M. and Tarunin, E. L., Secondary convective motions in a plane vertical fluid layer. Izv. Akad. Nauk SSSR, Mekhanika Zhidkosti i Gaza, №5, 1968.
12. Batchelor, G. K., Heat transfer by free convection across a closed cavity between vertical boundaries at different temperatures. Quart. Appl. Math., Vol. 12, p. 3, 1954.
13. Elder, J. W., Laminar free convection in a vertical slot. J. Fluid Mech., Vol. 23, p. 1, 1965.

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ON APPROXIMATIONS AND THE COARSENESS OF THE PARAMETER SPACE OF A DYNAMIC SYSTEM

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The equations of motion of dynamic systems used in modeling the behavior of engineering devices usually allow one to isolate not only the system parameters, but also the parameter-free (sinusoidal, polygonal, relay-type, etc.) normalized characteristics describing the individual elements of the device under consideration. The choice of such characteristics is always to some extent arbitrary, being dictated by: (a) the need to ensure adequate agreement between the behavior of the approximating characteristic and that of the true characteristic of the device, and (b) the need to obtain a system of equations amenable to investigation in sufficient detail.

Suitable choice of a characteristic (a suitable approximation) is a major phase in the construction of a usable model. The chosen characteristic is associated with a specific decomposition of the parameter space of the system into domains characterized by differing qualitative structures of the phase space. A natural requirement in constructing a convenient model by altering the approximation is that of preserving the qualitative structure of the parameter space decomposition as well as the structure of the phase space of the system in making the alteration. This makes it necessary to determine the degree to which it is possible to alter the characteristics of a system without essentially altering the general way in which the trajectories depend on the parameters, and also to ascertain the changes in the parameter space associated with changes in the characteristics.

1. Let us consider equations of the form

$$x^{\bullet} = P[x,\ y,\ F_i(x),\ \lambda_k], \qquad y^{\bullet} = Q[x,\ y,\ \psi_j(x),\ \lambda_k] \tag{1.1}$$

where $F_i(x)$ and $\psi_j(x)$ are the piecewise-continuous (and in a particular case analytic) characteristics of the system, and λ_k are the parameters.

Definition 1.1. We call the parameter space λ_k of system (1.1) "coarse" with respect to the class of characteristics $F_i(x)$ and $\psi_j(x)$ if the qualitative structure of the decomposition of the parameter space λ_k into domains of the same (or in some sense similar) structure of the decomposition of the phase space into trajectories remains unchanged for all the characteristics belonging to this class.

The problem of isolating the classes of characteristics with respect to which the parameter space λ_k of system (1.1) is coarse reduces to the problem of investigating the bifurcations [1] which can occur in the system upon alteration of the characteristics. If the replacement of one characteristic by another is not accompanied by the disappearance of any possible bifurcation or with the appearance of any new ones, then the system is "coarse" with respect to these characteristics in the above sense. The problem is a very difficult one in the general case, and there are no regular methods for its solution. The bifurcations which can occur in a system are amenable to investigation in differing degrees. The simplest of them are characterized by values of certain quantities at a point of the phase space (this category includes bifurcations of complex equilibrium states and bifurcations involved in estimates of the number of limit cycles arising from an equilibrium state of the focus or separatrix-loop type). Other bifurcations require information on the global behavior of trajectories and cannot be obtained by regular methods (this includes the very complex problems concerning the existence of saddle-to-saddle separatrices and the birth of double limit cycles from a trajectory condensation).

However, in certain cases such global estimates can be obtained by exploiting the specific properties of the equations under investigation or by using specially chosen comparison systems, whereupon the problem becomes completely solvable. A more frequent possibility is that of isolating classes of characteristics for which one can ensure the constancy of the decomposition of the parameter space into domains not of identical, but in some sense similar, structure.

For example, we can agree not to distinguish between domains of the parameter space associated with decompositions of the phase space which may differ by an even number of limit cycles. Such a formulation of the problem is often useful, since it expands the class of characteristics, extends the range of suitable choice of approximations, and therefore increases the opportunities of obtaining full information about all the most important characteristics of operation of the device as determined by the parameters. At the same time this approach serves to circumvent the often unsolvable problem of tracing the bifurcations associated with a double cycle.

2. Let us see how the above considerations apply to the equation

$$\varphi^{\cdot\cdot} + h\varphi^{\cdot} + F(\varphi) = \gamma \tag{2.1}$$

which occurs in certain problems of mechanics, electrical engineering, automatic phase control theory [2–4], etc., for various characteristics $F(\varphi)$.

Statement 2.1. If the function $F(\varphi)$ is differentiable, periodic with the period 2π, and piecewise-monotonic with two extrema in the period ($|\mathrm{extr} F(\varphi)| = 1$), and if it satisfies the condition

$$\int_0^{2\pi} F(\varphi)\, d\varphi = 0 \tag{2.2}$$

then the parameter space $\gamma > 0$, $h > 0$ is coarse with respect to the class of characteristics $F(\varphi)$.

The parameter space breaks down into three domains corresponding to the three possible coarse decompositions of the phase space φ, $\varphi^{\cdot}$ into trajectories. The various $F(\varphi)$ are associated merely with different dispositions of the bifurcation curve in the strip $0 < \gamma < 1$ of the parameter plane (Fig. 1).

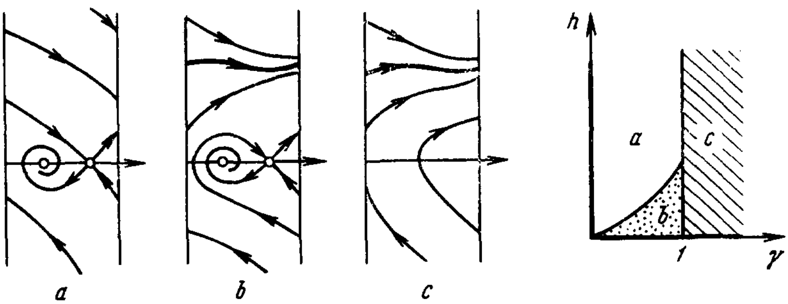

Fig. 1

Equation (2.1) is equivalent to the system

$$\frac{d\varphi}{dt} = y \equiv P, \qquad \frac{dy}{dt} = \gamma - hy - F(\varphi) \equiv Q \tag{2.3}$$

and therefore to the equation

$$\frac{dy}{d\varphi} = \frac{\gamma - hy - F(\varphi)}{y} \tag{2.4}$$

Let us consider (2.3) and (2.4) on the phase cylinder $-\pi \leqslant \varphi \leqslant \pi$ (the straight lines $\varphi = \pm\pi$ are identical). The two equilibrium states of (2.3) or the two singular points of (2.4) lie on the axis $y = 0$ and constitute a focus and a saddle for all characteristics of the class $F(\varphi)$. The character of the equilibrium states can be determined from the roots of the characteristic equation and depends on the sign of $F'(\varphi)$ at the point in question. The derivative $F'(\varphi)$ has different signs at the neighboring singular points. The focus is stable for $h > 0$. For $\gamma = 1$ the singular points merge, forming a singular point of the saddle-node type.

The Dulac criterion [5] enables us to formulate exhaustive statements about the bifurcations associated with limit cycles. Since the quantity $P_\varphi' + Q_y' \equiv -h$ does not change sign in the domain of the parameter space under consideration, there are no limit cycles surrounding the equilibrium state, and there cannot be more than one limit cycle girding the phase cylinder. Bifurcations associated with the sprouting of a limit cycle from a condensation of trajectories (i.e. with the birth of a double limit cycle) are an impossibility for the class of characteristics under consideration.

Condition (2.2) implies that a bifurcation associated with a separatrix loop can exist in the upper half-cylinder ($y > 0$) only. In fact, if there exists a closed contour consisting of the integral curves of Eq. (2.4), then

$$\int_0^{2\pi} [\gamma - hy(\varphi) - F(\varphi)]\, d\varphi \equiv \frac{1}{2} \int_0^{2\pi} d\,[y(\varphi)]^2 = 0$$

or, by virtue of (2.2),

$$\int_0^{2\pi} [\gamma - hy(\varphi)]\, d\varphi = 0$$

which is impossible for $y(\varphi) < 0$ and positive h and γ. Every trajectory of system (2.3) on the lower half-cylinder intersects the axis $y = 0$ (provided it is not the ω-separatrix of the saddle).

We must now establish the possibility of bifurcations associated with the sprouting of a limit cycle from a separatrix loop in the upper half-cylinder for the characteristics of the class under consideration.

To be specific, we choose the origin of the coordinate φ in such a way that the conditions $F(-\pi) = 0$, $F'(-\pi) < 0$ are fulfilled for the characteristic $F(\varphi)$ (this is always possible for system (2.3). We denote the other root of the equation $F(\varphi) = 0$ by φ_0.

Let us introduce the comparison system

$$\frac{d\varphi}{dt} = y, \quad \frac{dy}{dt} = \gamma - hy - \Phi(\varphi), \quad \Phi(\varphi) = \begin{cases} -1 & (-\pi < \varphi < \varphi_0) \\ 0 & (\varphi_0 < \varphi < \pi) \end{cases} \qquad (2.5)$$

The characteristic $\Phi(\varphi)$ lies below the characteristic $F(\varphi)$ of system (2.3). The vector field of system (2.5) is rotated relative to the vector field of system (2.3) by a positive angle on the upper half-cylinder.

The qualitative structures of the decomposition of the phase space and of the parameter space of comparison system (2.5) are readily obtainable. For $h > 0$ and $\gamma > 0$ system (2.5) has just one structure of the decomposition of the phase space into trajectories. All of the trajectories proceed from infinity towards a limit cycle lying in the strip $(\gamma + 1) / h < \varphi < \gamma / h$ on the upper half-cylinder.

Let us trace the behavior of the trajectories of system on the upper half-cylinder, using what we know about the behavior of the trajectories of system (2.3) for $h = 0$ and comparison system (2.5).

We begin by considering by the decomposition into trajectories for system (2.3) in the case $h = 0$ $(0 < \gamma < 1)$. Equation (2.4) is integrable. Two singular points lie on the phase cylinder; these are $y = 0, \varphi = \varphi_1$ (a center) and $y = 0, \varphi = \varphi_2$ (a saddle), where φ_1, φ_2 are the roots of the equation $\gamma - F(\varphi) = 0$ $(\varphi_2 > \varphi_1)$. The equation of the separatrices passing through the saddle is

$$y^2 = 2\gamma(\varphi - \varphi_2) - 2\Phi(\varphi), \qquad \Phi(\varphi) = \int_{\varphi_2}^{\varphi} F(\varphi)\, d\varphi \qquad (2.6)$$

The function $\Phi(\varphi)$ is periodic with the period 2π, piecewise-monotonic with two extrema per period at the points $\varphi = -\pi(\varphi = \pi)$ and $\varphi = \varphi_0$ and vanishing at the points φ_2 and $\varphi_{2'}(-\pi < \varphi_{2'} < \varphi_2)$.

The equation

$$\gamma(\varphi - \varphi_2) - \Phi(\varphi) = 0$$

always has a unique simple root $\varphi = \varphi^*(-\pi < \varphi_{2'} < \varphi^* < \varphi_2)$ for $\gamma \neq 0$ $(0 < \gamma < 1)$ in addition to the double root $\varphi = \varphi_2$. Hence, for $0 < \gamma < 1$ separatrix (2.6) forms a loop which surrounds the equilibrium state $\varphi = \varphi_1$. We note that the α-separatrix of the saddle on the upper half-cylinder cannot return to the same saddle; it winds on the cylinder out to infinity (infinity is stable for $h = 0$).

Only for $\gamma = 0$ do we have $\varphi^* = \varphi_{2'} = -\pi$, whereupon the separatrix forms a loop girding the cylinder. For any $\gamma \neq 0$ $(0 < \gamma < 1)$ we can always choose an h so small that the α-separatrix of the saddle on the upper half-cylinder also winds on the cylinder. Since infinity for system (2.3) is unstable for $h > 0$, this fact implies the existence of a stable limit cycle girding the cylinder for small h (this limit cycle is unique by virtue of the Dulac criterion).

For large h the structure of the decomposition of the phase space into trajectories can be determined with the aid of comparison system (2.5). On the upper half-cylinder the representing point which moves along a trajectory of system (2.3) from left to right intersects the trajectories of system (2.5) downward. Let $y = \eta$ be the point of intersection of the ω-separatrix of the saddle on the upper half-cylinder with the straight line $\varphi = \varphi_1$. If we choose h in such a way that the upper edge of the strip containing the limit cycle of system (2.5) lies below the straight line $y = \eta$, so that $(\gamma + 1)/h < \eta$, then the ω-separatrix of the saddle on the upper half-cylinder enters the domain (above the strip containing the limit cycle of system (2.5)) filled by the trajectories which intersect the trajectories of system (2.5) downward. In this case system (2.3) cannot have a limit cycle. Such a choice of h for $0 < \gamma < 1$ is always possible, since the vector field rotates clockwise with increasing h, which means that η increases.

Comparison of the structures of the phase-space decomposition for small and large h implies the existence for $0 < \gamma < 1$ of a bifurcation curve for whose points the α- and ω-separatrices on the upper half-cylinder form a loop girding the cylinder. This curve is single-valued in h for $0 < \gamma < 1$, since monotonic variation of h is accompanied by monotonic rotation of the vector field on the upper half-cylinder. The bifurcation curve begins at the point $h = \gamma = 0$.

For $\gamma = 1$ there is a complex saddle-node point on the axis $y = 0$. There exists a unique value $h = h_0$ for which the α- and ω-separatrices of the saddle-node form a loop girding the cylinder. For $0 < h < h_0$ there exists a stable limit cycle which girds the cylinder; for $h_0 < h < \infty$ all of the trajectories have the saddle-node as their limit point and no cycles exist. A unique stable limit cycle exists for $\gamma > 1$ and any h, since there are no limit points and since infinity is unstable in this case.

The class $F(\varphi)$ of characteristics of system (2.3) for which the parameter space λ_h is coarse can be made to include polygonal and discontinuous characteristics by posing a restricted problem and not distinguishing between structures of the decomposition of the phase space into trajectories which are similar in a certain sense. It turns out that despite the (sometimes considerable) differences in the local behavior of the trajectories in the phase space, the decomposition of the space on conversion to such characteristics remains unchanged in many respects and retains the general pattern of possible bifurcations in the parameter space under consideration. We can, for example, regard as similar, and not distinguish between, a focus (or node) and a "composite focus", and a saddle and a "composite saddle". Figure 2 shows: (a) a focus composed of ordinary trajectories; (b) a saddle composed of ordinary trajectories (the singular point is not an equilibrium state); (c) a saddle composed of two different analytic saddles (the singular point is an equilibrium state).

The structure of the decomposition of the phase space in the neighborhood of these composite singular points is in the ordinary sense identical to the decomposition in the neighborhood of the analytic focus and saddle, but the behavior of the trajectories with respect to t differs markedly.

We can go still further in some cases by not drawing a distinction between a trajectoty continuum and a single trajectory, provided their behavior as $t \to \infty$ or $t \to -\infty$ is in some sense similar.

Let us consider some examples of such identifications. Upon shifting of the phase plane along the broken line, the curve l in Fig. 3a becomes identical with the continuum of curves l' which merge into a slip segment and a portion of the curve l (Fig. 3b).

An isolated singular point of the focus (or node) type can be identified either with an attraction or repulsion segment (Fig. 4a), or with a bounded domain filled with closed trajectories and playing the role of an attraction or repulsion element for other trajectories (Fig. 4b).

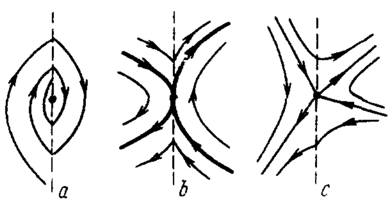

Fig. 2

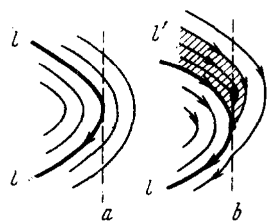

Fig. 3

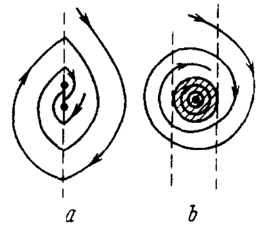

Fig. 4

This identification of a singular point with linear or two-dimensional attraction or repulsion elements naturally give rise to analogs of the bifurcations associated with the birth of a limit cycle from a singular point upon a change in stability.

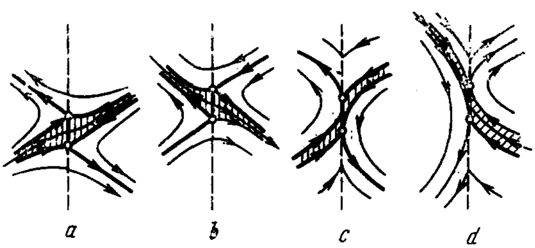

Fig. 5

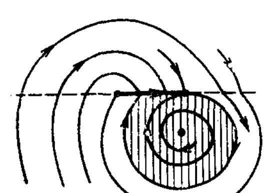

Fig. 6

Figure 5 shows some possible cases of identification of the separatrix of an ordinary analytic saddle with the trajectory continuum of a composite saddle: Fig. 5a is a saddle composed of two shifted analytic saddles with an attraction segment for the trajectory continuum; Fig. 5b is a saddle composed of analytic saddles with a repulsion segment of the trajectory continuum; Fig. 5c is a saddle composed of ordinary trajectories with an attraction segment for the trajectory continuum (there are no equilibrium states at the ends of the rest segment); Fig. 5d is a saddle with a repulsion segment composed of ordinary trajectories. This does not exhaust the range of possible identifications. For example, one can identify a stable limit cycle with a limit cycle containing a slip segment which the representing point traveling in the trajectory continuum reaches after a finite time (Fig. 6).

The inclusion of polygonal and discontinuous characteristics in the class $F(\varphi)$ yields considerably simpler equations amenable to complete qualitative investigation and enabling one to obtain the equations of the boundaries in which bifurcations in the parameter space take place. Moreover, the qualitative results of investigations to within the above identifications coincide with the results corresponding to the analytical characteristics of the class $F(\varphi)$.

For example, the class $F(\varphi)$ for Eq. (2.1) can be extended by means of the polygonal characteristics (Fig. 7a)

$$F_1(\varphi) = \begin{cases} 2(\varphi+\pi)/(\pi+\lambda)-1, & -\pi \leqslant \varphi \leqslant \lambda \\ -2(\varphi-\lambda)/(\pi-\lambda)+1, & \lambda \leqslant \varphi \leqslant \pi \end{cases} \tag{2.7}$$

or the relay-type characteristics (Fig. 7b)

$$F_2(\varphi) = \begin{cases} -\pi/(\pi+\lambda), & -\pi < \varphi < \lambda \\ \pi/(\pi-\lambda), & \lambda < \varphi < \pi \end{cases} \tag{2.8}$$

where λ is an "internal parameter" of the family of characteristics [6].

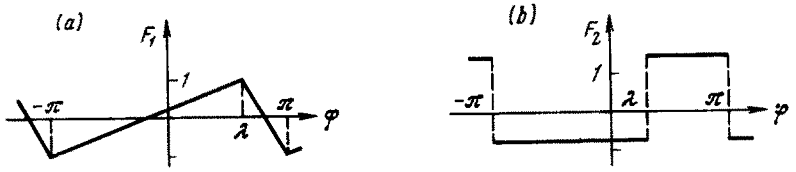

Fig. 7

It is easy to find the equations of the curves on which the bifurcations for these characteristics lie. For example, we can readily establish that for characteristic (2.7) the bifurcation curve in the plane γ, h passes through the origin and through the point $(1, 2\sqrt{2/(\pi+\lambda)})$ at which it joins the vertical segment of the boundary.

The equation of the bifurcation curve is especially simple for $\lambda = \pi$ (when characteristic (2.7) becomes discontinuous),

$$\gamma = \frac{e^H - 1}{e^H + 1}, \qquad H = \frac{\pi^{3/2} h}{\sqrt{4 - \pi h^2}}$$

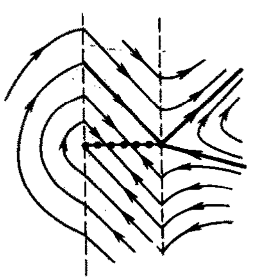

Fig. 8

For $\lambda \neq \pi$ and $\gamma < 1$ in the phase space there are no singular points on the line of matching. For $\lambda = \pi$ and $\gamma < 1$ the two singular points are a focus and a saddle (on the line of matching) composed of ordinary trajectories. For $\gamma = 1$ there is a complex singular point on the line of matching; this point vanishes for $\gamma > 1$.

In the case of characteristic (2.8) the merging of the bifurcation curve with the vertical segment of the boundary occurs at the point $\gamma = \pi/(\pi - \lambda)$, $h = h_0$, where h_0 is the root of the equation

$$2\pi^2\left[1 - \exp\left(-\frac{h^2(\pi^2-\lambda^2)}{\pi}\right)\right] = h^2(\pi-\lambda)^2(\pi+\lambda)$$

For $\gamma < \pi/(\pi-\lambda)$ there are two singular points (a composite focus and a composite saddle) composed of ordinary trajectories on the lines of matching in the phase space. For $\gamma = \pi/(\pi - \lambda)$ there arises a special formation (Fig. 8) similar to a saddle-node. This formation contains the attraction segment $y = 0$, $\lambda < \varphi < \pi$ and vanishes with increasing γ (the index of the closed curve which encloses the attraction segment and adjacent trajectories is equal to zero).

The parameter space γ, h of Eq. (2.1) is coarse with respect to the class of characteristics $F(\varphi)$, $F_1(\varphi)$ and $F_2(\varphi)$ if we identify the attraction or repulsion segments similar, in the above sense, with each other.

3. Let us consider some examples of systems with a more complex decomposition of the parameter space which is coarse with respect to some class of characteristics. First, let us take the system of equations (describing the self-oscillations of a synchronous motor [7, 8])

$$dy/dt = D - \psi_1(\varphi) - [A + B\psi_2(\varphi) - C\psi_1(\varphi)]y, \quad d\varphi/dt = y \qquad (3.1)$$

where $\psi_1(\varphi)$ (odd) and $\psi_2(\varphi)$ (even) are periodic with the periods 2π and π, respectively, for three types of characteristics: the analytic characteristic $\psi_1 = \sin\varphi$, $\psi_2 = \cos 2\varphi$ (Fig. 9a), a polygonal characteristic (Fig. 9b), and a relay-type characteristic (Fig. 9c).

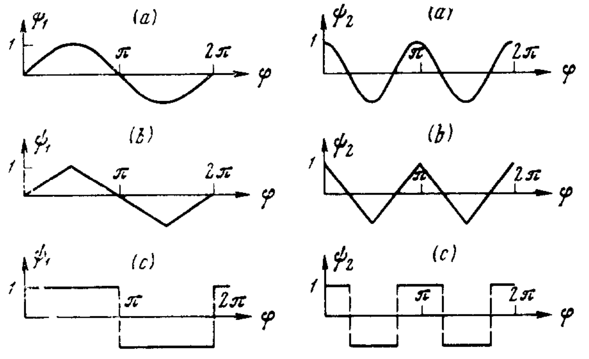

Fig. 9

Let us introduce the small positive parameter μ, setting $D = \mu T$, $A = \mu\alpha$, $B = \mu\beta$, $C = \mu\gamma$. The analytical characteristics turn out to be given by the equations

$$dy/dt = -\sin\varphi + \mu[T - (\alpha + \beta\cos 2\varphi - \gamma\sin\varphi)y], \quad d\varphi/dt = y \qquad (3.2)$$

System (3.2) has two equilibrium states, a focus and a saddle. For small μ the focus is stable if $\alpha + \beta > 0$ and unstable if $\alpha + \beta < 0$.

The structure of the decomposition of the phase space into trajectories is determined by the character of the singular points, by the character and disposition of the limit cycles, and by the behavior of the separatrices. Upon introduction of the small parameter the system can be investigated conveniently with the aid of Pontriagin's theorem [9] which enables us to investigate the behavior of the limit cycles of nearly-Hamiltonian systems and to determine their existence domains.

From $\mu = 0$ system (3.2) has the integral $H(\varphi, y) \equiv {}^1/_2\, y^2 - \cos\varphi = h$. The values of the constant h from the interval $-1 < h < 1$ are associated with closed integral curves which surround an equilibrium state (of the center type); the values from the interval $1 < h < \infty$ are associated with integral curves girding the phase cylinder.

For $h = 1$ the saddle separatrices form a loop girding the cylinder.

If we rewrite system (3.2) as

$$d\varphi/dt = H_y' + \mu p(\varphi,y) \qquad dy/dt = -H_\varphi' + q(\varphi,y) \tag{3.3}$$

then the values of the constant h which isolate the roots of the conservative system whose neighborhoods contain the limit cycles of system (3.3) on the upper and lower half-cylinders for small μ are the roots of the equations

$$\psi_1(h) = 0, \quad \psi_2(h) = 0$$

where

$$\psi_1(h) = \int_0^{2\pi} q\,d\varphi - p\,dy = \int_0^{2\pi} [T - (\alpha + \beta\cos 2\varphi - \gamma\sin\varphi)\,y]\,d\varphi =$$

$$= \beta\int_0^{2\pi} [\nu - (\sigma - 2\sin^2\varphi)\sqrt{2(\cos\varphi + h)}]\,d\varphi =$$

$$= \beta\left[2\pi\nu + \left(\frac{128}{15}\frac{k^4 - k^2 + 1}{k^5} - \frac{8\sigma}{k}\right)E\left(\frac{\pi}{2}, k\right) + \frac{64}{15}\frac{3k^2 - k^4 - 2}{k^5}F\left(\frac{\pi}{2}, k\right)\right]$$

$$\psi_2(h) = \beta\left[2\pi\nu - \left(\frac{128}{15}\frac{k^4 - k^2 + 1}{k^5} - \frac{8\sigma}{k}\right)E\left(\frac{\pi}{2}, k\right) - \frac{64}{15}\frac{3k^2 - k^4 - 2}{k^5}F\left(\frac{\pi}{2}, k\right)\right]$$

Here $F(^1/_2\,\pi, k)$ and $E(^1/_2\pi, k)$ are total elliptic integrals of the first and second kind, and

$$\nu = T/\beta, \quad \sigma = (\alpha + \beta)/\beta, \quad k^2 = 2/(h + 1) \quad (1 < h < \infty)$$

The values of the constant h_0 which isolate the curves C_{h_0} of the conservative system surrounding the equilibrium state are the roots of the equation

$$\psi_3(h) = 0$$

where

$$\psi_3(h) = \iint_{C_{h_0}} (p'_\varphi + q'_y)\,d\varphi\,dy = -\iint_{C_{h_0}} (\alpha + \beta\cos 2\varphi - \gamma\sin\varphi)\,d\varphi\,dy =$$

$$= -2\sqrt{2}\beta\int_{-\varphi_0}^{\varphi_0} (\sigma - 2\sin^2\varphi)\sqrt{\cos\varphi + h}\,d\varphi =$$

$$= -16\beta\{[\sigma - {}^{16}/_{15}(\varkappa^4 - \varkappa^2 + 1)]\,E(^1/_2\pi, \varkappa) + [\sigma(\varkappa^2 - 1) -$$
$$- {}^8/_{15}(3\varkappa^2 - \varkappa^4 - 2)]\,F(^1/_2\pi, \varkappa)\}$$

$$\varkappa^2 = {}^1/_2(h + 1) \quad (-1 < h < 1), \qquad \varphi_0 = \arccos(-h)$$

The roots of the equations $\psi_1(h) = 0$, $\psi_2(h) = 0$, $\psi_3(h) = 0$ depend on the two parameters σ and ν. We can decompose the parameter plane σ, ν in such a way that the domains of the decomposition correspond to the various possible distributions of the roots of the equations under consideration. Each distribution is associated with a specific structure of the decomposition of the phase space into trajectories. The following set of conditions (each condition is associated with some curve in the plane σ, ν) defines all of the bifurcations possible in system (3.3):

(1) $\psi_3(-1) = 0$, (2) $\psi_1(\infty) = 0$ or $\psi_2(\infty) = 0$
(3) $\psi_3(1) = 0$, (4) $\psi_1(1) = 0$, (5) $\psi_2(1) = 0$
(6) $\psi_1(h) = 0$ and $\psi_1'(h) = 0$, (7) $\psi_2(h) = 0$ и $\psi_2'(h) = 0$
(8) $\psi_3(h) = 0$ and $\psi_3'(h) = 0$,

To be specific, let us set $\beta > 0$, $\nu > 0$ and consider the upper half-plane σ, ν (for $\nu < 0$ we obtain a decomposition of the parameter space which is symmetric to the domain $\nu > 0$ with respect to the axis σ). In this case the equation $\psi_1(h) = 0$ has not more than a single root, and $\psi_2(h) = 0$, $\psi_3(h) = 0$ not more than two roots apiece. The above conditions are associated with the following equations of boundary curves and bifurcations:

1. $\sigma = 0$. An unstable limit cycle arises out of the focus with increasing σ.
2. $\sigma = 1$. A stable limit cycle arises out of $+\infty$ with increasing σ. An unstable limit cycle arises out of $-\infty$ with decreasing σ.
3. $\sigma = {}^{16}/_{15} = 1.066\ldots$ A stable limit cycle surrounding the equilibrium state arises from the saddle separatrix with increasing σ (since $P_\varphi' + Q_y' \equiv \equiv -\mu\beta\sigma < 0$) at the saddle).
4. $2\pi\nu - 8\sigma + {}^{128}/_{15} = 0$. A stable limit cycle arises out of the separatrix loop on the upper half-plane with decreasing σ.
5. $2\pi\nu + 8\sigma - {}^{128}/_{15} = 0$. An unstable limit cycle arises out of the separatrix loop on the lower half-cylinder with increasing σ (if $-\beta\mu\sigma > 0$) ; a stable limit cycle arises out of the separatrix loop with decreasing σ (if $-\mu\beta\sigma < 0$)
6. $\nu > 0$. No curve exists (there cannot be two limit cycles on the upper half-cylinder for $\nu > 0$).
7. Setting $\psi_2(h) \equiv \beta[2\pi\nu + \sigma\Phi_1(h) - \Phi_2(h)] = 0$, we can write the parametric equations of the curve as

$$\sigma = \frac{\Phi_2'(h)}{\Phi_1'(h)}, \qquad \nu = \frac{\Phi_2(h)\Phi_1'(h) - \Phi_2'(h)\Phi_1(h)}{2\pi\Phi_1(h)}$$

The curve passes between the points $A(0,\ {}^{128}/_{30}\ \pi^{-1} = 1.36)$ and $B(1, 0)$. As σ increases the double limit cycle which has arisen out of the trajectory condensation on the lower half-cylinder splits into two limit cycles (of which the lower is unstable and the upper stable).

8. Eliminating σ from $\psi_3(h) \equiv -16\ \beta\ [\Psi_1(h)\sigma - \Psi_2(h)] = 0$ and $\psi_3'(h) = 0$, we obtain an equation for determining h. The equation $\Psi_2'\Psi_1 - \Psi_2\Psi_1' = 0$ has the single root $h = 0.86$ which corresponds to $\sigma = 1.09$. The double limit cycle surrounding the equilibrium state vanishes with increasing σ.

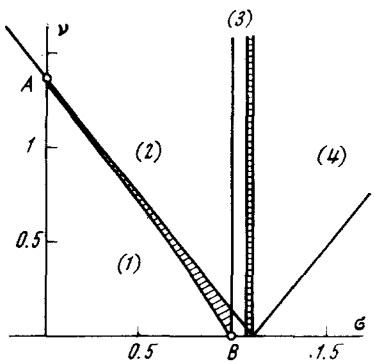

Fig. 10

The decomposition of the parameter space σ, ν into domains associated with various qualitative structures of the decomposition of the phase space is shown in Fig. 10. The two slender shaded areas represent the domains for whose points the phase space contains two limit cycles.

In the case of polygonal and discontinuous characteristics system (3.1) can be investigated in similar fashion by the small-parameter method. The sufficient conditions of applicability of Pontriagin's formula [9] to systems with piecewise-analytic right sides are formulated in [10].

Let $H(x, y), = h$ be the family of closed curves dependent on the parameter h and composed of the segments $H_i(x, y) = h$ in the intervals $x_i < x < x_{i+1}$. We assume that the functions $H_i(x, y)$ are analytic in both of their arguments. The system

$$x^{\cdot} = H_y' + \mu p(x, y), \qquad y^{\cdot} = -H_x' + \mu q(x, y)$$

where $p(x, y)$, $q(x, y)$ are analytic in each interval $x_i < x < x_{i+1}$ and where μ is a small parameter, has (for $\mu \neq 0$) a single limit cycle in the neighborhood of the closed curve C_{h_0}, where h_0 is the root of the equation

$$\psi(h_0) \equiv \int_{C_{h_0}} q(x, y)\,dx - p(x, y)\,dy = 0 \tag{3.4}$$

and $\psi'(h_0) \neq 0$ if $\partial H / \partial y$ is continuous at the points of matching $x = x_i$. The limit cycle is stable if $\psi'(h_0) < 0$ and unstable if $\psi'(h_0) > 0$.

In a special case the curves C_{h_0} can gird the phase cylinder.

System (3.1) with polygonal characteristics (Fig. 9b) and a small parameter μ [11](*) has the following integral for $\mu = 0$:

$$H(\varphi, y) = \frac{y^2}{2} + \begin{Bmatrix} -(\varphi+\pi)^2/\pi \\ \varphi^2/\pi - \pi/2 \\ -(\varphi-\pi)^2/\pi \end{Bmatrix} = h \quad \begin{matrix} (-\pi < \varphi < -\pi/2) \\ (-\pi/2 < \varphi < \pi/2) \\ (\pi/2 < \varphi < \pi) \end{matrix} \tag{3.5}$$

The closed curves of family (3.5) surround the singular point for $-{}^1/_2\pi < h < 0$ and gird the phase cylinder for $0 < h < \infty$. The equilibrium states for $\mu \neq 0$ are shifted off the line of matching. These points are $O_1({}^1/_2\,\mu T\pi, 0)$ (a focus) and $O_2(\pi - {}^1/_2\,\mu T\pi, 0)$ (a saddle).

The equations of the boundaries in the parameter space σ and ν listed in the same order as for (3.2) are

(1) $\sigma = 0$, (2) $\sigma = 1$, (3) $\sigma = {}^4/_3\sqrt{2}(1 + {}^1/_4\pi)^{-1} = 1.056\ldots$

(4) $2\pi\nu - {}^1/_2\sqrt{2}\pi^{3/2}(1 + {}^1/_4\pi)\sigma + {}^4/_3\pi^{3/2} = 0$

(5) $2\pi\nu + {}^1/_2\sqrt{2}\pi^{3/2}(1 + {}^1/_4\pi)\sigma - {}^4/_3\pi^{3/2} = 0$

(6) nonexistent for $\nu > 0$

(7) the curve passes between the points $A(0,\ {}^2/_3\sqrt{\pi} = 1.26\ldots)$ and $B(1.0)$

(8) $\sigma = 1{,}07\ldots$

The decompositions of the phase space and parameter space for polygonal characteristics remain qualitatively identical to the decompositions for the characteristics of Fig. 9a. The slender domains for whose points the phase space contains two limit cycles

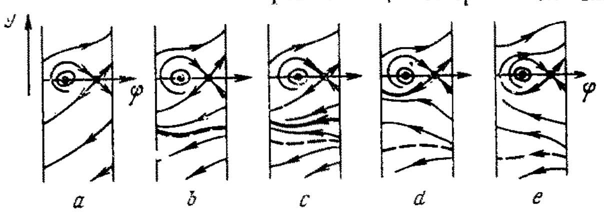

Fig. 11

*) The author of [11] points out the errors made in [8] (and repeated in [12]) concerning the existence of an unstable limit cycle.

are likewise preserved, even though their dimensions are slightly altered. Figures 11a and 11e show the decompositions of the cylindrical phase space for domains (1) and (2) of Fig. 10, respectively.

In the case of polygonal characteristics (Fig. 9b) the qualitatively equivalent decompositions of the phase space (Figs. 11a and 11e) correspond to the domains of the parameter space situated (as in Fig. 10) in the strip $0 < \sigma < 1$. For both of the above approximations of Eq. (3.1) domains (1) and (2) in the parameter space are separated by a narrow strip whose points are associated with two limit cycles in the phase space. This strip does not vanish with a change in the approximation despite its narrowness (its maximum width of 0.015 for $\sigma = 1$ decreases very rapidly as $\sigma \to 0$, since the bifurcation curve between the points A and B osculates the straight line which is the lower boundary of domain (2) at the point A). The coarseness of the parameter space with respect to changes of the characteristic with retention of the "slender" elements is not self-evident and is related to the preservation of the properties of the bifurcations with the disappearance and appearance of a separatrix loop for various approximations. These properties are determined by the sign of the quantity $P_\varphi' + Q_y'$ for the saddle [1].

For a fixed σ it is possible to pass from domain (1) into domain (2) by traveling in the direction of increasing ν. The decomposition of the phase space shown on a strip of width 2π (Fig. 11a) becomes the decomposition of Fig. 11e. At $\nu = \nu_0$ (this value is unique by virtue of the monotonic variation of the direction of the vector field with monotonic variation of ν) the α- and ω-separatrices of the saddle on the lower half-cylinder must form a loop girding the cylinder.

However, a loop cannot give rise to the unstable limit cycle shown in Fig. 11e, since the saddle quantity given to within terms of the order μ^2 by the expression $P_\varphi' + Q_y' = -\mu\beta\sigma$ for both approximations (Figs. 9a and 9b) is negative in the interval $0 < \sigma < 1$ $(\beta > 0)$, so that the limit cycle must become a separatrix loop with increasing ν. This implies that a double limit cycle arises when ν has increased to the value $\nu = \nu_0$.

This cycle then divides into two cycles (a stable upper cycle and an unstable lower cycle), and the stable limit cycle becomes a separatrix loop which vanishes with further increases in ν and generates the decomposition shown in Fig. 11b (the successive transitions are shown in Figs. 11a–11e).

The above considerations enable us to define the class of characteristics for which domains (1) and (2) are necessarily separated by a two-cycle domain. All of the above statements apply almost verbatim to the conditions of existence of a thin strip with a phase space containing two limit cycles (surrounding the equilibrium state) and separating domains (3) and (4).

A change in the characteristic is generally associated with a shift of the bifurcation curves on the parameter plane and of their points of intersection. If the parameter plane contains points of intersection of more than two bifurcation curves (and therefore points of contact of more than four domains), then the neighborhoods of such points can alter the qualitative structure of the decomposition of the parameter plane with a change of the characteristic.

Preservation of the structure of the parameter plane decomposition at these points requires the imposition of more rigid conditions on the class of characteristics which do not alter the structure of the parameter plane decomposition. An example of such a point for the parameter plane σ, ν for the characteristics of Figs. 9a and 9b is point A, which is the meeting point of five domains. The unchanging character of the qualitative structure of the decomposition of the parameter plane of system (3.1) in the case of the characteristics of Figs. 9a and 9b is due to the fact that the quantity $P_\varphi' + Q_y'$ for the focus and saddle, which is constant to within quantities of the order of μ^2, has the same value for both approximations, and a change in the sign of σ is associated not only with the sprouting of a cycle out of the singular point, but also with the change in the character of the bifurcations for a separatrix loop. This condition is not satisfied in the case of relay-type characteristics.

Let us consider system (3.1) with relay characteristics (Fig. 9c) and a small parameter (the analysis which follows was carried out by I. A. Nepomniashchaia). For $\mu = 0$ this system has the integral

$$H(\varphi, y) = \begin{Bmatrix} y^2/2 - \varphi \\ y^2/2 + \varphi \end{Bmatrix} = h \quad \begin{matrix} (-\pi < \varphi < 0) \\ (0 < \varphi < \pi) \end{matrix} \tag{3.6}$$

The closed curves of family (3.6) surround the singular point for $0 < h < \pi$ and gird the phase cylinder for $\pi < h < \infty$. The system has singular points on the lines of matching, namely a composite focus at $O_1(0, 0)$ and a saddle composed of ordinary trajectories at $O_2\ (\pi, 0)$.

The functions $\psi_1(h)$, $\psi_2(h)$ and $\psi_3(h)$ are especially simple in the case of relay-type characteristics. Let us write out the expression for $\psi_3(h)$, which has some interesting properties. We have

$$\psi_3(h) = -\,{}^8/_3 \sqrt{2}\,[\sigma h^{3/2} - 2m(h - {}^1/_4\pi)^{3/2} + 2n(h - {}^3/_4\pi)^{3/2}] \tag{3.7}$$

where

$$m = n = 0, \quad \text{if} \quad 0 \leqslant h \leqslant {}^1/_4\,\pi$$

$$m = 1,\ n = 0, \quad \text{if} \quad {}^1/_4\pi \leqslant h \leqslant {}^3/_4\,\pi$$

$$m = n = 1, \quad \text{if} \quad {}^3/_4\,\pi \leqslant h \leqslant \pi$$

Figure 12 shows curves (3.7) for several σ in the plane $h\psi$. For $\sigma = 0$ the function $\psi_3(h)$ for $0 \leqslant h \leqslant {}^1/_4\,\pi$ coincides with a segment of h-axis, and therefore has a continuum of roots.

The passage of σ through zero corresponding to the sequence of changes of the qualitative structures shown in Fig. 13 is the analog of the bifurcation corresponding to the sprouting of an unstable

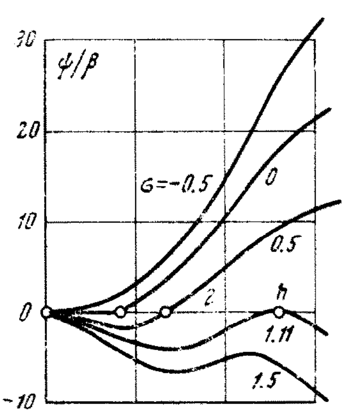

Fig. 12

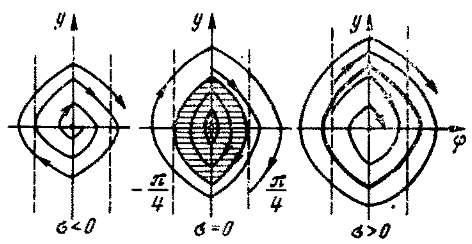

Fig. 13

limit cycle from the singular point. The limit cycle arises out of the boundary of the domain filled with closed curves (*). (See Note at the foot of the next page).

Let us write out the equations of the boundaries on the plane $\sigma\nu$

(1) $\sigma = 0$, (2) $\sigma = 1$, (3) $\sigma = {}^1/_4(3\sqrt{3} - 1) = 1.049\ldots$

(4) $2\pi\nu - {}^1/_8\sqrt{2}\pi^{1/2}(4\sigma + 1 - 3\sqrt{3}) = 0$

(5) $2\pi\nu + {}^1/_8\sqrt{2}\pi^{1/2}(4\sigma + 1 - 3\sqrt{3}) = 0$

(6) nonexistent for $\nu > 0$

(7) the curve passes between the points

$$A\left[\sqrt{3} - 1,\ \sqrt{{}^1/_2\pi}\left(1 - \frac{\sqrt{3}}{3}\right)\right] \text{ and } B(1, 0)$$

(8) $'\sigma = {}^4/_{13}\sqrt{13} = 1.11\ldots$

The decompositions of the phase space of system (3.1) with relay-type characteristics are not qualitatively equivalent to the corresponding decompositions for analytic and polygonal characteristics in all domains of the parameter space; in those cases where differences arise, however, the decompositions are nevertheless similar and admit of identification in the above sense. The difference in bifurcations is on the straight line $\sigma = 0$ (the birth of an unstable limit cycle from the boundary of some domain containing a singular point which accompanies a change in the sign of σ).

The decomposition of the parameter space σ, ν for the system with relay-type characteristics differs from that shown in Fig. 10 in the position of the curve AB. Point A does not lie on the axis $\sigma = 0$. The curve AB osculates the boundary of domain (2) at point A for $\sigma = \sqrt{3} - 1$. As ν increases in the interval $0 < \sigma < \sqrt{3} - 1$, we can have a direct transition from the decomposition of Fig. 11a to the decomposition of Fig. 11e which bypasses the two-cycle domain; this direct transition occurs through the birth of an unstable limit cycle out of the separatrix loop girding the cylinder. The character and relative disposition of the other bifurcation curves remain unchanged under replacement of polygonal or analytic characteristics by relay-type characteristics.

4. Let us consider the system (describing the symmetric flight of an aircraft in a vertical plane at a constant angle of attack [13])

$$d\varphi / dt = y^2 - \cos\varphi, \qquad dy / dt = y(\alpha - \beta y^2 - \sin\varphi) \tag{4.1}$$

and introduce [14] the small parameter $\beta = \mu$, $\alpha = k\mu$. For $\mu = 0$ the system has the integral

$$H(\varphi, y) = {}^1/_3 y^3 - y\cos\varphi = h \tag{4.2}$$

---

*) For $\sigma = 0$ the composite equilibrium state on the line of matching is a "center to within quantities of the order of μ^2". With allowance for terms of order μ^2 the strip $-{}^1/_4\pi < \varphi < {}^1/_4\pi$ contains "slowly winding" or "slowly unwinding" spirals. This can be shown, for example, by constructing the succession function on the half-line $y \geqslant 0$ of the line of matching. It is of the form

$$y_2 = y_1 - {}^4/_3 T\gamma y_0^2\mu^2 + (\ldots)\mu^3 + \ldots \qquad (y_0 \text{ is a parameter})$$

The function ψ_3^* describing bifurcations in the neighborhood of the singular point with allowance for terms of the order of μ^2 cannot be determined from (3.4), but is obtainable from the so-called second approximations.

The closed curves of family (4.2) surround the equilibrium state for $-{}^2/_3 < h < 0$ and gird the phase cylinder for $0 < h < \infty$. For a small μ the system has three equilibrium states: The saddles $O_1(-{}^1/_2\,\pi, 0)$ and $O_2({}^1/_2\,\pi, 0)$ and the focus $O_3[(k-1)\mu, 1 + {}^1/_2(k^2-1)\mu^2]$ (which becomes a center for $\mu = 0$). The phase space is cylindrical. By virtue of the physical meaning of the variables and parameters we need consider only the upper half-cylinder ($y = 0$ is the integral curve) and the positive values of the parameters.

A distinguishing feature of the decomposition of the phase space into trajectories consists in the fact that the parameter values associated with the appearance of a saddle-to-saddle separatrix are also associated with the simultaneous appearance of two closed contours consisting of the separatrices of a saddle on a cylinder and segments of the axis φ, i.e. a contour surrounding the equilibrium state and a contour girding the phase cylinder. As the parameters vary, the contours consisting of saddle separatrices give rise either to a limit cycle surrounding the equilibrium position or a limit cycle girding the phase cylinder. The function $\psi(h)$ whose roots determine the structure of the decomposition into trajectories can therefore be written in standard form for all cycles, i.e.

$$\psi(h) = \iint_{C_{h_0}} (k - 3y^2)\,dy\,d\varphi \qquad \text{or} \qquad \psi(h) = \int_{-\pi}^{\pi} y(k - y^2)\,d\varphi$$

for $-{}^2/_3 < h < 0$ or $0 < h < \infty$, respectively; both of these cases are covered by the expression

$$\psi(h) = 2\int_{e_2}^{e_1} \frac{(k - y^2)(2y^3 + 3h)}{\sqrt{9y^2 - (y^3 - 3h)^2}}\,dy \qquad (4.3)$$

Here e_1 and $e_2 < e_1$ are either the positive roots of the equation $y^3 - 3y = 3h$ if $-{}^2/_3 < h < 0$ or the positive roots of the equations $y^3 - 3y = 3h$ and $y^3 + 3y = 3h$, respectively, if $h > 0$. The definition of the function $\psi(h)$ is complemented by its limiting values for $h = -{}^2/_3$ and $h = 0$.

Analysis shows that $\psi(h) = 0$ cannot have more than two roots in the interval $0 < h < \infty$ and more than one root for $-{}^2/_3 < h < 0$.

All of the possible bifurcations in system (4.1) with a small parameter are covered by the following set of conditions (each of which is associated with a specific value of k).

1. $\psi(-{}^2/_3) = 0$, $\psi'(-{}^2/_3) = 0$. An unstable limit cycle arises out of the equilibrium state as k decreases.
2. $\psi(0) = 0$. An unstable limit cycle girding the cylinder arises from a separatrix loop with decreasing k, and an unstable limit cycle surrounding the equilibrium state arises with increasing k.
3. $\psi(h^*) = 0$, $\psi'(h^*) = 0$ $(h^* > 0)$. The double limit cycle vanishes with decreasing k. The double limit cycle splits into two limit cycles (the upper one stable, the lower one unstable) with increasing k.

The above conditions are associated with the following values of the parameter k:

$$(1)\ k = 3, \qquad (2)\ k = \Gamma^4({}^1/_4)\,/\,8\,\pi^2 = 2.188, \qquad (3)\ k = 2{,}05$$

Figure 14 shows the form of the functions $\psi(h)$ for several values of k. Figure 15 shows the decomposition in the plane of small parameters α, β of the phase space into trajectories with domains characterized by differing qualitative structures. The points lying within the shaded strip are associated with the two limit cycles in the phase space.

Let us consider system (4.1) containing a small parameter with $\cos\varphi$ approximated by a sawtooth function and $\sin\varphi$ by a relay function (the analysis which follows was carried out by I. S. Sharova).

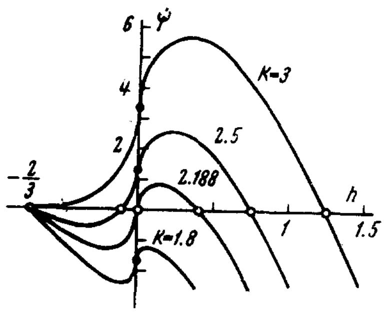

Fig. 14

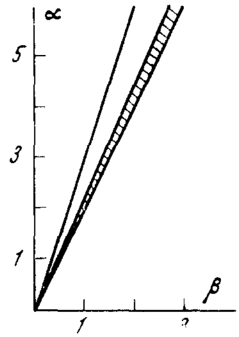

Fig. 15

For $\mu = 0$ the system has the integral

$$H(\varphi, y) = \frac{y^3}{3} - y + \begin{Bmatrix} 2y\varphi/\pi \\ -2y\varphi/\pi \end{Bmatrix} = h \quad \begin{matrix} (-\pi < \varphi < 0) \\ (0 < \varphi < \pi) \end{matrix} \tag{4.4}$$

for which the derivative $\partial H/\partial y$ is continuous at the lines of matching $\varphi = 0$ and $\varphi = \pm\pi$.

The closed curves of family (4.4) surround the equilibrium state of the "composite center" type for $-{}^2/_3 < h < 0$ and gird the phase cylinder for $0 < h < \infty$. The equilibrium states $O_1(-{}^1/_2\pi, 0)$ and $O_2({}^1/_2\pi, 0)$ are saddles. The function $\psi(h)$ in this case becomes

$$\psi(h) = \frac{\pi}{3}\int_{e_2}^{e_1}\left[\frac{3kh}{y} + 2ky^2 - 3hy - 2y^4\right]dy \tag{4.5}$$

and e_1 and e_2 have the same values as in the previous case.

Analysis reveals the identity of the behavior and properties of the functions $\psi(h)$ for the initial and approximating systems with respect to the dependence of the roots on the parameter k. The corresponding bifurcation values of k for the approximating system are $k = 3$, ${}^9/_5$, 1.65. The parameter space of the system differs from that shown in Fig. 15 only by a slight shift of the shaded strip corresponding to two-cycle systems.

The identity of the compositions of the phase space for the initial and approximating systems is due primarily to the preservation of the properties of the bifurcations associated with the saddle separatrices, since the saddle quantity does not change upon transition to the approximating system (at the saddle $P_\varphi + Q_y' = k\mu$ for both systems).

5. It is a well-known fact that the results of qualitative investigations by the small-parameter method often remain valid for systems in which the parameters are known in advance to be nonsmall. This "small-μ miracle" is due to the conditions of coarseness of the parameter space with respect to the class of characteristics including characteristics with a small parameter μ which need not necessarily assume small values (here the parameter μ plays the role of an internal parameter in the class of characteristics dependent on both variables).

If a dynamic system with certain parameter values (on the set A_0) has a (plane or cylindrical) phase space with a continuum of closed curves (surrounding an equilibrium state of the center type or in the form of closed curves on a cylinder), if for other parameter values (on the sets A_i) the system can have special noncoarse elements (a noncoarse focus, a double limit cycle , or saddle-to-saddle separatrices), and if the parameter space is not specially degenerate, then the set A_0 intersects the sets A_i (*). For this reason even a small neighborhood of the set A_0 must contain a set of bifurcations and domains associated with various decompositions of the phase space which are possible for the given system. The possibility of other bifurcations, e.g. of bifurcations related to changes in the number of singular points of the system, often requires a substantial rather than a small change in the parameters.

Let us return to Eqs. (4.1), no longer assuming that the parameters α and β are small. A change in the number of equilibrium states of system (4.1) occurs for $\alpha > 1$. Let us consider the domain $\alpha < 1$ in which the number of equilibrium states is the same as that for small μ. The simplest bifurcations associated with a limit cycle can be found and have the same character as for small α and β. The appearance of a stable limit cycle out of infinity occurs as β increases from zero (this is evident directly from Eq. (4.1), since infinity changes from stable to unstable when the sign of β is altered). An unstable limit cycle grows out of the equilibrium state along the curve

$$\alpha - 3\beta \frac{\alpha\beta + \sqrt{1+\beta^2-\alpha^2}}{1+\beta^2} = 0 \qquad (4.6)$$

(the corresponding boundary in Fig. 15 is the tangent to curve (4.6) at the origin). The quantity $P_\varphi' + Q_y' = \alpha = k\mu$ for nonsmall μ does not alter the sign and preserves the constant character of the bifurcations associated with the saddle separatrices. It is only the bifurcations associated with the double limit cycle about which we do not have sufficient information to draw inferences. Knowledge of the other bifurcations enables us to draw limited conclusions concerning the domain of existence of systems with two limit cycles.

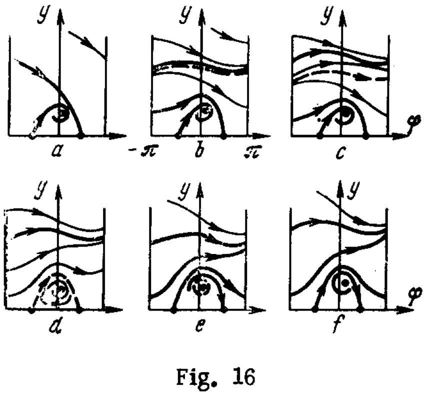

Fig. 16

The disposition of saddle separatrices for large values of the parameter β is shown in Fig. 16a (this follows directly from the disposition of the principal isoclines for sufficiently large β). There are no limit cycles girding the cylinder. The disposition of separatrices for small β is shown in Fig. 16f (for $\beta = 0$ the α-separatrix of the saddle goes out to infinity, as can be shown by the method of [15]; for small β a stable limit cycle arises out of infinity). The vector field rotates monotonically with decreasing β, so that

*) If $H(x, y) = h$ is a family of curves containing a continuum of closed trajectories, if the equation is written as

$$\frac{dy}{dx} = \frac{-H_x' + q(x, y)}{H_y' + p(x, y)}$$

and if the parameter space is complete in the sense that none of the coefficients occurring in $p(x, y)$ and $q(x, y)$ is a constant making a cut in the complete space, then this statement is almost self-evident.

there is a single value of β_0 for any fixed α_0 for which the α- and ω-separatrices form a loop. The set of points α_0, β_0 forms a continuous curve which intersects the strip $0 < \alpha < 1$.

However, a stable limit cycle girding the cylinder cannot arise out of the loop, since $P_\varphi' + Q_y' = \alpha > 0$ at the saddle.

Only an unstable limit cycle can sprout from, or contact into, the loop; this can occur as β decreases to the value $\beta = \beta_0$ only if a double limit cycle girding the cylinder arises out of the condensation of trajectories.

This limit cycle then splits into two limit cycles (the upper one is stable, the lower one unstable), and the unstable limit cycle can become a separatrix loop which vanishes with further decreases in β and generates an unstable limit cycle surrounding the equilibrium state. The unstable limit cycle contracts into the equilibrium state and vanishes for the value of β satisfying condition (4.6).

The above description of the changes in the qualitative structure of the decomposition of the phase space into trajectories which accompany changes in β enable us to postulate the necessary existence of a domain with a phase space containing two limit cycles which gird the cylinder; it also enables us to trace the analogous sequence of bifurcations dependent on β as in the case of a small μ. However, the sequence of structures of the decompositions of the phase space into trajectories shown in Fig. 16 and proved rigorously for the case of small μ can be identified with the corresponding structures for nonsmall μ to within an even number of limit cycles only.

The logical possibility of such a discrepancy has not been eliminated, and the coarseness of the parameter space must be understood in the restricted sense formulated at the beginning of the present paper. In this sense the above description proves the coarseness of the parameter space with respect to a transition from small to nonsmall μ in a fairly wide strip ($0 < \alpha < 1$) of the parameter space α, β.

BIBLIOGRAPHY

1. Andronov, A. A., Leontovich, E. A., Gordon, I. I. and Maier, A.G., The Theory of Bifurcations of Dynamic Systems on a Plane. Moscow, "Nauka", 1967.
2. Giger, A., Ein Grenzproblem einer technisch wichtigen nichtlinearen Differentialgleichung. Z. angew. Math. Phys., Vol. 7, Fasc. 2, 1956.
3. Tricomi, F., Integrazione di un equazione differenziale presantatasi in electrotechnica. Ann. Scuola Norm. Super., Pisa, Ser. 2, Vol. 2, №1, 1933.
4. Kapranov, M. V., An approximate method for computing the capture band of automatic phase control. In: Second Inter-VUZ Conference on the Theory and Methods of Calculation of Nonlinear Electrical Circuits, Vol. 2, Tashkent, 1963.
5. Dulac, H., Recherche des cycles limités. C. R. Acad. Sci., t. 204, №23, 1937.
6. Ternikova, N. P., A study of a certain differential equation of automatic phase control. Tr. Gor'kovsk. inst. inzh. vodn. transp. №94, 1968.
7. Dreyfus, L., Einführung in die Theorie der selbsterregten Schwingungen synchroner Maschinen. Electrotechnik und Maschinenbau, 29, H. 16, s. 323-342, 1911.
8. Vlasov, N. P., Self-oscillations of a synchronous motor. Uch. zap. Gor'kovsk. univ. Vol. 12, №3, 1939.

9. Pontriagin, L. S., Über Autoschwingungssysteme, die den Hamiltonschen nahe liegen. Phys. Zeit. Sowjetunion, Bd. 6, H. 1-2, S. 25-28, 1934.
10. Serebriakova, N. N., Periodic solutions of second order dynamic systems close to piece-wise Hamiltonian systems. PMM Vol. 33, №5, 1969.
11. Gersht, E. N., A qualitative study of a certain differential equation of the theory of electrical machines. Izv. Akad. Nauk SSSR, Mekhan. i mashinostr. №1, 1964.
12. Minorsky, N., Introduction to Nonlinear Mechanics, USA, Ann. Arbor. J. W. Edwards, 1947.
13. Bautin, N. N., The Behavior of Dynamic Systems Near the Boundaries of the Stability Domain. Leningrad-Moscow, Gostekhizdat, 1949.
14. Bautin, N. N., On the nearly fugoid longitudinal motions of an aircraft. Uch. zap. Gor'kovsk. univ. №13, 1947.
15. Ikonnikov, E., On the dynamics of symmetrical flight of an aeroplane. Technical Physics of the USSR. Vol. 4, №6, pp. 433-437, 1937.

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OPTIMIZATION OF PROCESSES WITH DIFFERENCE ARGUMENTS

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Optimization of a process with difference arguments is considered. Necessary optimality conditions are obtained in the form of maximum principle. The problem is reduced to a boundary-value problem for a system of ordinary differential equations with no difference arguments. This is performed by a special transformation.

The damping of vibrations of a string is considered as an example.

1. Some important problems in mathematical physics such as the damping of one-dimensional vibrational processes (see Example) can be reduced to the following optimal problem.

For the process $x(t) = (x_1(t),\ldots, x_n(t))$ with the values $x \in X \subset E_n$ for each $t \in [0, k t_k]$ the process being described on the portions $[st_k, (s+1)t_k]$ $(s = 0,\ldots, k-1)$ by the equations

$$\frac{dx}{dt}\bigg|_{st_k+\tau} = \varphi^s(\tau, z^+, z^-) \qquad (\tau \in [0, t_k];\ s = 0, \ldots, k-1) \tag{1.1}$$

with the boundary conditions

$$f_j(x_0^\circ, \ldots, x_0^{k-1}, x_k^\circ, \ldots, x_k^{k-1}) = 0 \qquad (j = 1, \ldots, q;\ q < 2nk) \tag{1.2}$$

it is required to find a control $u(t) = (u_1(t),\ldots, u_r(t))$ with the values $u \in U \subset E_r$ for each $t \in [0, kt_k]$ which minimizes the functional

$$J = f_0(x_0^\circ, \ldots, x_0^{k-1}, x_k^\circ, \ldots, x_k^{k-1}) \tag{1.3}$$

where $x(t)$ is a continuous time vector-function and $u(t)$ is a piece-wise continuous time vector-function on $[st_k, (s+1)t_k]$; $\varphi^s = (\varphi_1^s, \ldots, \varphi_n^s)$, $f_0, \ldots, f_q$ are continuous and twice continuously differentiable ; t_k is a specified value.